

# BASES FOR THE GLOBAL WEYL MODULES OF $\mathfrak{sl}_n$ OF HIGHEST WEIGHT $m\omega_1$

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ABSTRACT. We utilize a theorem of B. Feigin and S. Loktev to give explicit bases for the global Weyl modules for the map algebras  $\mathfrak{sl}_n \otimes A$  of highest weight  $m\omega_1$ . These bases are given in terms of specific elements of  $U(\mathfrak{sl}_n \otimes A)$  acting on the highest weight vector.

## 1. INTRODUCTION

Let  $\mathfrak{g}$  be a simple finite dimensional complex Lie algebra. For the loop algebras,  $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ , the global Weyl modules were introduced by Chari and Pressley, [4]. Feigin and Loktev extended these global Weyl modules to the case where the Laurent polynomials above were replaced by the coordinate ring of a complex affine variety, [5]. Chari, Fourier and Khandai then generalized this definition to the map algebras,  $\mathfrak{g} \otimes A$ , where  $A$  is a commutative, associative complex unital algebra, [3]. Feigin and Loktev also gave an isomorphism, which explicitly determines the structure of the global Weyl modules for the map algebras of  $\mathfrak{sl}_n$  of highest weight  $m\omega_1$ , [5].

The goal of this work is to use the structure isomorphism given by Feigin and Loktev to give nice bases for the global Weyl modules for the map algebras of  $\mathfrak{sl}_n$ ,  $\mathfrak{sl}_n \otimes A$ , of highest weight  $m\omega_1$ . These bases will be given in terms of specific elements of  $U(\mathfrak{sl}_n \otimes A)$  acting on the highest weight vector. This was done in [2] in the case  $n = 2$ , but the case  $n > 2$  has not previously appeared in the literature.

## 2. PRELIMINARIES

**2.1. The Structure of  $\mathfrak{sl}_n$ .** Recall that  $\mathfrak{sl}_n$  is the Lie algebra of all complex traceless matrices. The Lie bracket is the commutator bracket given by  $[A, B] = AB - BA$ .

Given any matrix  $[b_{i,j}]$  define  $\varepsilon_k([b_{i,j}]) := b_{k,k}$ . For  $i \in \{1, \dots, n-1\}$  define  $\alpha_i := \varepsilon_i - \varepsilon_{i+1}$ . Define

$$R^\pm := \left\{ \pm(\alpha_i + \dots + \alpha_j) \mid 1 \leq i < j \leq n-1 \right\}$$

to be the positive and negative roots respectively, and define  $R = R^+ \cup R^-$  to be the set of roots. Let  $e_{i,j}$  be the  $n \times n$  matrix with a one in the  $i$ th row and  $j$ th column and zeros in every other position. Define  $h_i := h_{\alpha_i} = e_{i,i} - e_{i+1,i+1}$ , for  $i \in \{1, \dots, n-1\}$ . Then  $\mathfrak{h} := \text{span}\{h_i \mid 1 \leq i \leq n\}$  is a Cartan sub-algebra of  $\mathfrak{sl}_n$ . Given  $\alpha = \alpha_i + \dots + \alpha_j \in R^+$  define  $x_\alpha := e_{i,j}$  and  $x_{-\alpha} := e_{j,i}$ . Then  $\{h_i, x_{\pm\alpha} \mid 1 \leq i \leq n-1, \alpha \in R\}$  is a Chevalley basis for  $\mathfrak{sl}_n$ . Given  $i \in \{1, \dots, n-1\}$ , define  $x_i := x_{\alpha_i} = e_{i,i+1}$ ,  $x_{-i} := x_{-\alpha_i} = e_{i+1,i}$ . Note that, for all  $1 \leq i \leq n-1$ ,  $\text{span}\{x_{-i}, h_i, x_i\} \cong \mathfrak{sl}_2$ .

Define nilpotent sub-superalgebras  $\mathfrak{n}^\pm := \text{span}\{x_\alpha \mid \alpha \in R^\pm\}$  and note that  $\mathfrak{sl}_n = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ .

Define the set of fundamental weights  $\{\omega_1, \dots, \omega_{n-1}\} \subset \mathfrak{h}^*$  by  $\omega_i(h_j) = \delta_{i,j}$  for all  $i, j \in \{1, \dots, n-1\}$ . Define  $P^+ := \text{span}_{\mathbb{Z}_{\geq 0}}\{\omega_1, \dots, \omega_{n-1}\}$  to be the set of dominant integral weights.

**2.2. Map Algebras and Weyl Modules.** For the remainder of this work fix a commutative, associative complex unital algebra  $A$ . Define the map algebra of  $\mathfrak{sl}_n$  to be  $\mathfrak{sl}_n \otimes A$  with Lie bracket given by linearly extending the bracket

$$[z \otimes a, w \otimes b] = [z, w] \otimes ab$$

for all  $z, w \in \mathfrak{sl}_n$  and  $a, b \in A$ .

Define  $\mathbf{U}(\mathfrak{sl}_n \otimes A)$  to be the universal enveloping algebra of  $\mathfrak{sl}_n \otimes A$ .

As in [3] we define the global Weyl model for  $\mathfrak{sl}_n \otimes A$  of highest weight  $\lambda \in P^+$  to be the module generated by a vector  $w_\lambda$ , called the highest weight vector, with relations:

$$(x \otimes a)w_\lambda = 0, \quad (h \otimes 1)w_\lambda = \lambda(h)w_\lambda, \quad (x_{-i} \otimes 1)^{\lambda(h_i)+1}.w_\lambda = 0$$

for all  $a \in A$ ,  $x \in \mathfrak{n}^+$ ,  $h \in \mathfrak{h}$ , and  $1 \leq i \leq n-1$ .

**2.3. Multisets.** Given any set  $S$  define a multiset of elements of  $S$  to be a multiplicity function  $\chi : S \rightarrow \mathbb{Z}_{\geq 0}$ . Define  $\mathcal{F}(S) := \{\chi : S \rightarrow \mathbb{Z}_{\geq 0} : |\text{supp } \chi| < \infty\}$ . For  $\chi \in \mathcal{F}(S)$  define  $|\chi| := \sum_{s \in S} \chi(s)$ . Notice that  $\mathcal{F}(S)$  is an abelian monoid under function addition. For  $\psi, \chi \in \mathcal{F}(S)$ ,  $\psi \subseteq \chi$  if  $\psi(s) \leq \chi(s)$  for all  $s \in S$ . Define  $\mathcal{F}(\chi)(S) := \{\psi \in \mathcal{F}(S) \mid \psi \subseteq \chi\}$ . In the case  $S = A$  the  $S$  will be omitted from the notation. So that  $\mathcal{F} := \mathcal{F}(A)$  and  $\mathcal{F}(\chi) = \mathcal{F}(\chi)(A)$ .

If  $\psi, \chi \in \mathcal{F}$  with  $\psi \in \mathcal{F}(\chi)$  we define  $\chi - \psi$  by standard function subtraction. Also define  $\pi : \mathcal{F} - \{0\} \rightarrow A$  by

$$\pi(\psi) := \prod_{a \in A} a^{\psi(a)}$$

and extend  $\pi$  to  $\mathcal{F}$  by setting  $\pi(0) = 1$ . Define  $\mathcal{M} : \mathcal{F} \rightarrow \mathbb{Z}$  by

$$\mathcal{M}(\psi) := \frac{|\psi|!}{\prod_{a \in A} \psi(a)!}$$

For all  $\psi \in \mathcal{F}$ ,  $\mathcal{M}(\psi) \in \mathbb{Z}$  because if  $\text{supp } \psi = \{a_1, \dots, a_k\}$  then  $\mathcal{M}(\psi)$  is the multinomial coefficient

$$\binom{|\psi|}{\psi(a_1), \dots, \psi(a_k)}$$

For  $s \in S$  define  $\chi_s$  to be the characteristic function of the set  $\{s\}$ . Then for all  $\chi \in \mathcal{F}(S)$

$$\chi = \sum_{s \in S} \chi(s) \chi_s$$

**2.4. The Symmetric Tensor Space.** Given any vector space  $W$ , there is an action of the symmetric group  $S_k$  on  $\underbrace{W \otimes W \otimes \dots \otimes W}_{k\text{-times}}$  given by

$$\sigma(w_1 \otimes w_2 \otimes \dots \otimes w_k) = v_{\sigma^{-1}(1)} \otimes v_{\sigma^{-1}(2)} \otimes \dots \otimes v_{\sigma^{-1}(k)} \text{ where } v_1, \dots, v_k \in W.$$

For any vector space  $W$ , define its  $k$ th symmetric tensor space

$$S^k(W) = \text{span} \left\{ \sum_{\sigma \in S_k} \sigma(w_1 \otimes \cdots \otimes w_k) \mid w_1, \dots, w_k \in W \right\}$$

Define  $V \cong \mathbb{C}^n$  to be an  $\mathfrak{sl}_n$ -module via left matrix multiplication, and write the basis as  $v_1 := (1, 0, \dots, 0)$ , and for  $i \in \{1, \dots, n+m-1\}$ ,  $v_{i+1} := x_{-i}v_i$ . Then  $V \otimes A$  is an  $\mathfrak{sl}_n \otimes A$  module under the action  $(z \otimes a)(w \otimes b) = zw \otimes ab$ .

Given  $\varphi_1, \dots, \varphi_n \in \mathcal{F}$  with  $k := \sum_{i=1}^n |\varphi_i|$  define

$$w(\varphi_1, \dots, \varphi_n) := \bigotimes_{a_1 \in \text{supp } \varphi_1} (v_1 \otimes a_1)^{\otimes \varphi_1(a_1)} \otimes \cdots \otimes \bigotimes_{a_n \in \text{supp } \varphi_n} (v_n \otimes a_n)^{\otimes \varphi_n(a_n)} \in (V \otimes A)^{\otimes k}$$

and

$$v(\varphi_1, \dots, \varphi_n) := \sum_{\sigma \in S_k} \sigma(w(\varphi_1, \dots, \varphi_n)) \in S^k(V \otimes A).$$

We will need the following theorem of Feigin and Loktev, which is Theorem 6 in [5].

**Theorem 2.4.1** (Feigin–Loktev, 2004). *For all  $m \in \mathbb{N}$   $W_A(m\omega_1) \cong S^m(V \otimes A)$  via the map given by*

$$w_{m\omega_1} \mapsto (v_1 \otimes 1)^{\otimes m}.$$

We will also need the following lemma.

**Lemma 2.4.2.** *Let  $\mathbf{B}$  be a basis for  $A$ . Then the set*

$$\mathfrak{B} := \left\{ v(\varphi_1, \dots, \varphi_n) \mid \varphi_1, \dots, \varphi_n \in \mathcal{F}(\mathbf{B}), \sum_{i=1}^n |\varphi_i| = m \right\}$$

*is a basis for  $S^m(V \otimes A)$ .*

*Proof.*  $\mathfrak{B}$  spans  $S^m(V \otimes A)$  because  $\mathbf{B}$  spans  $A$  and  $v_1, \dots, v_n$  spans  $V$ .  $\mathfrak{B}$  is linearly independent because the set

$$\{(v_{j_1} \otimes b_1) \otimes \cdots \otimes (v_{j_m} \otimes b_m) \mid j_1, \dots, j_m \in \{1, \dots, n\}, b_1, \dots, b_m \in \mathbf{B}\}$$

is a basis for  $(V \otimes A)^{\otimes m}$  and hence is linearly independent.  $\square$

Given  $k \in \mathbb{N}$  define  $\Delta^{k-1} : \mathbf{U}(\mathfrak{sl}_n \otimes A) \rightarrow \mathbf{U}(\mathfrak{sl}_n \otimes A)^{\otimes k}$  by extending the map  $\mathfrak{sl}_n \otimes A \rightarrow \mathbf{U}(\mathfrak{sl}_n \otimes A)^{\otimes k}$  given by

$$\Delta^{k-1}(z \otimes a) = \sum_{j=0}^{k-1} 1^{\otimes j} \otimes (z \otimes a) \otimes 1^{\otimes k-1-j}$$

Note that  $\Delta^{k-1}(1) = 1^{\otimes k}$  not  $k1^{\otimes k}$ .

Since  $V \otimes A$  is a  $\mathbf{U}(\mathfrak{sl}_n \otimes A)$  module,  $(V \otimes A)^{\otimes m}$  is a left  $\mathbf{U}(\mathfrak{sl}_n \otimes A)$ -module with  $u$  acting as  $\Delta^{m-1}(u)$  followed by coordinatewise module actions. Moreover  $S^m(V \otimes A)$  is a submodule under this action. Thus  $S^m(V \otimes A)$  is a left  $\mathbf{U}(\mathfrak{sl}_n \otimes A)$ -module under this  $\Delta^{m-1}$  action.

2.5. For all  $i = 1, \dots, n-1$  and  $\chi, \varphi \in \mathcal{F}$  recursively define  $q_i(\varphi, \chi) \in \mathbf{U}(\mathfrak{sl}_n \otimes A)$  as follows

$$\begin{aligned} q_i(0, 0) &:= 1 \\ q_i(0, \chi) &:= -\frac{1}{|\chi|} \sum_{0 \neq \psi \in \mathcal{F}(\chi)} \mathcal{M}(\psi)(h_i \otimes \pi(\psi)) q_i(0, \chi - \psi) \\ q_i(\varphi, \chi) &:= -\frac{1}{|\varphi|} \sum_{\psi \in \mathcal{F}(\chi)} \sum_{d \in \text{supp } \varphi} \mathcal{M}(\psi)(x_{-i} \otimes d\pi(\psi)) q_i(\varphi - \chi d, \chi - \psi) \end{aligned}$$

Given  $\varphi_1, \dots, \varphi_n \in \mathcal{F}$ , define

$$q(\varphi_1, \dots, \varphi_n) := q_{n-1}(\varphi_n, \varphi_{n-1}) q_{n-2}((|\varphi_n| + |\varphi_{n-1}|)\chi_1, \varphi_{n-2}) \dots q_2 \left( \left( \sum_{j=3}^n |\varphi_j| \right) \chi_1, \varphi_2 \right) q_1 \left( \left( \sum_{k=2}^n |\varphi_k| \right) \chi_1, \varphi_1 \right)$$

**Remark 2.5.1.** Note that the  $q_i(0, \chi)$  coincide with the  $p_i(\chi)$  defined in [1].

### 3. MAIN THEOREM

The main result of this work is the theorem stated below.

**Theorem 3.0.2.** Given a basis  $\mathbf{B}$  for  $A$  and  $m \in \mathbb{Z}_{>0}$ , the set

$$\left\{ q(\varphi_1, \dots, \varphi_n) w_{m\omega_1} \mid \varphi_1, \dots, \varphi_n \in \mathcal{F}(\mathbf{B}), \sum_{i=1}^n |\varphi_i| = m \right\}$$

is a basis for  $W_A(m\omega_1)$ .

The proof of this theorem will be given after several necessary lemmas and propositions.

#### 3.1. Necessary Lemmas and Propositions.

**Proposition 3.1.1.** For all  $k \in \mathbb{N}$   $\Delta^k = (1^{\otimes k-1} \otimes \Delta^1) \circ \Delta^{k-1}$ .

*Proof.* The case  $k = 1$  is trivial. For  $k \geq 2$  and  $u \in \mathbf{U}(\mathfrak{sl}_n \otimes A)$  we have

$$\begin{aligned}
(1^{\otimes k-1} \otimes \Delta^1) (\Delta^{k-1}(u)) &= (1^{\otimes k-1} \otimes \Delta^1) \left( \sum_{j=0}^{k-1} 1^{\otimes j} \otimes u \otimes 1^{\otimes k-1-j} \right) \\
&= (1^{\otimes k-1} \otimes \Delta^1) \left( \sum_{j=0}^{k-2} 1^{\otimes j} \otimes u \otimes 1^{\otimes k-1-j} + 1^{\otimes k-1} \otimes u \right) \\
&= \sum_{j=0}^{k-2} 1^{\otimes j} \otimes u \otimes 1^{\otimes k-2-j} \otimes \Delta^1(1) + 1^{\otimes k-1} \otimes \Delta^1(u) \\
&= \sum_{j=0}^{k-2} 1^{\otimes j} \otimes u \otimes 1^{\otimes k-2-j} \otimes 1 \otimes 1 + 1^{\otimes k-1} \otimes (u \otimes 1 + 1 \otimes u) \\
&= \sum_{j=0}^{k-2} 1^{\otimes j} \otimes u \otimes 1^{\otimes k-j} + 1^{\otimes k-1} \otimes u \otimes 1 + 1^{\otimes k-1} \otimes 1 \otimes u \\
&= \sum_{j=0}^k 1^{\otimes j} \otimes u \otimes 1^{\otimes k-j} \\
&= \Delta^k(u)
\end{aligned}$$

□

Given  $\chi \in \mathcal{F}$  and  $k \in \mathbb{N}$  define

$$\text{comp}_k(\chi) = \left\{ \psi : \{1, \dots, k\} \rightarrow \mathcal{F}(\chi) \mid \sum_{j=1}^k \psi(j) = \chi \right\}$$

**Lemma 3.1.2.** *For all  $i \in \{1, \dots, n-1\}$*

$$\Delta^{k-1}(q_i(\varphi, \chi)) = \sum_{\substack{\psi \in \text{comp}_k(\chi) \\ \phi \in \text{comp}_k(\varphi)}} q_i(\phi(1), \psi(1)) \otimes \cdots \otimes q_i(\phi(k), \psi(k))$$

*Proof.* This can be proven by induction on  $k$ . The case  $k = 1$  is trivial. In the case  $k = 2$  the lemma becomes

$$\Delta^1(q_i(\varphi, \chi)) = \sum_{\substack{\psi \in \mathcal{F}(\chi) \\ \phi \in \mathcal{F}(\varphi)}} q_i(\phi, \psi) \otimes q_i(\varphi - \phi, \chi - \psi)$$

This can be proven by induction on  $|\varphi|$ . For  $k > 2$  use Proposition 3.1.1. The details in the  $\mathfrak{sl}_2$  case can be found in [2]. This can be extended to the  $\mathfrak{sl}_n$  case via the injection  $\Omega_i : \mathfrak{sl}_2 \otimes A \rightarrow \mathfrak{sl}_n \otimes A$  given by

$$\Omega_i(x^- \otimes a) = x_{-i} \otimes a, \quad \Omega_i(h \otimes a) = h_i \otimes a, \quad \Omega_i(x^+ \otimes a) = x_i \otimes a$$

For all  $i \in \{1, \dots, n-1\}$  and  $a \in A$ .

□

**Lemma 3.1.3.** *For all  $\varphi, \chi \in \mathcal{F}$  with  $|\varphi| + |\chi| > 1$  and all  $i \in \{1, \dots, n-1\}$   $q_i(\varphi, \chi)(v_i \otimes 1) = 0$ .*

*Proof.* Assume that  $\varphi = 0$ . This case will proceed by induction on  $|\chi| > 1$ . If  $|\chi| = 2$  (so that  $\chi = \{a, b\}$  for some  $a, b \in A$ ) we have

$$\begin{aligned} q_i(0, \{a, b\})(v_i \otimes 1) &= [(h_i \otimes a) \otimes (h_i \otimes b) - (h_i \otimes ab)](v_i \otimes 1) \\ &= (h_i \otimes a) \otimes (v_i \otimes b) - (v_i \otimes ab) \\ &= (v_i \otimes ab) - (v_i \otimes ab) \\ &= 0 \end{aligned}$$

For the next case assume that  $|\chi| > 2$  then

$$q_i(0, \chi)(v_i \otimes 1) = -\frac{1}{|\chi|} \sum_{\emptyset \neq \psi \in \mathcal{F}(\chi)} \mathcal{M}(\psi)(h_i \otimes \pi(\psi)) q_i(\chi - \psi)(v_i \otimes 1) = 0$$

by induction.

Now assume that  $|\varphi| = 1$  (or  $\varphi = \chi_b$  for some  $b \in A$ ). Then

$$\begin{aligned} q_i(\chi_b, \chi)(v_i \otimes 1) &= - \sum_{\psi \in \mathcal{F}(\chi)} \mathcal{M}(\psi)(x_{-i} \otimes b\pi(\psi)) q_i(0, \chi - \psi)(v_i \otimes 1) \\ &= -\mathcal{M}(\chi)(x_{-i} \otimes b\pi(\chi))(v_i \otimes 1) - \sum_{a \in \text{supp } \chi} \mathcal{M}(\chi - \chi_a)(x_{-i} \otimes b\pi(\chi - \chi_a)) q_i(0, \chi_a)(v_i \otimes 1) \\ &= -\mathcal{M}(\chi)(v_{i+1} \otimes b\pi(\chi)) - \sum_{a \in \text{supp } \chi} \mathcal{M}(\chi - \chi_a)(x_{-i} \otimes b\pi(\chi - \chi_a))(-h_i \otimes a)(v_i \otimes 1) \\ &= -\mathcal{M}(\chi)(v_{i+1} \otimes b\pi(\chi)) + \sum_{a \in \text{supp } \chi} \mathcal{M}(\chi - \chi_a)(x_{-i} \otimes b\pi(\chi - \chi_a))(v_i \otimes a) \\ &= -\mathcal{M}(\chi)(v_{i+1} \otimes b\pi(\chi)) + \sum_{a \in \text{supp } \chi} \mathcal{M}(\chi - \chi_a)(v_{i+1} \otimes b\pi(\chi)) \\ &= -\mathcal{M}(\chi)(v_{i+1} \otimes b\pi(\chi)) + \sum_{a \in \text{supp } \chi} \frac{(|\chi| - 1)!}{\prod_{c \in \text{supp } (\chi - \chi_a)} (\chi - \chi_a)(c)!} (v_{i+1} \otimes b\pi(\chi)) \\ &= -\mathcal{M}(\chi)(v_{i+1} \otimes b\pi(\chi)) + \sum_{a \in \text{supp } \chi} \frac{(|\chi| - 1)!}{\prod_{\substack{c \in \text{supp } \chi \\ c \neq a}} \chi(c)! (\chi(a) - 1)!} (v_{i+1} \otimes b\pi(\chi)) \\ &= -\mathcal{M}(\chi)(v_{i+1} \otimes b\pi(\chi)) + \sum_{a \in \text{supp } \chi} \frac{\chi(a)}{|\chi|} \mathcal{M}(\chi)(v_{i+1} \otimes b\pi(\chi)) \\ &= -\mathcal{M}(\chi)(v_{i+1} \otimes b\pi(\chi)) + \mathcal{M}(\chi)(v_{i+1} \otimes b\pi(\chi)) \\ &= 0 \end{aligned}$$

Finally assume that  $|\varphi| > 1$ . Then

$$\begin{aligned}
q_i(\varphi, \chi)(v_i \otimes 1) &= -\frac{1}{|\varphi|} \sum_{\psi \in \mathcal{F}(\chi)} \sum_{d \in \text{supp } \varphi} \mathcal{M}(\psi)(x_{-i} \otimes d\pi(\psi)) q_i(\varphi - \chi_d, \chi - \psi)(v_i \otimes 1) \\
&= -\frac{1}{|\varphi|} \sum_{\psi \in \mathcal{F}(\chi)} \sum_{d \in \text{supp } \varphi} \mathcal{M}(\psi) \\
&\quad \left( -\frac{1}{|\varphi| - 1} \sum_{\psi_1 \in \mathcal{F}(\chi - \psi)} \sum_{d_1 \in \text{supp } (\varphi - \chi_d)} \mathcal{M}(\psi_1) \right. \\
&\quad \left. (x_{-i} \otimes d\pi(\psi))(x_{-i} \otimes d_1\pi(\psi_1)) q_i(\varphi - \chi_d - \chi_{d_1}, \chi - \psi - \psi_1) \right) (v_i \otimes 1) \\
&= 0
\end{aligned}$$

because at least two  $x_{-i}$  terms act on a single  $v_i$  as 0. □

**Lemma 3.1.4.** *For all  $i \in \{1, \dots, n-1\}$  and  $\varphi, \chi \in \mathcal{F}$  with  $|\varphi| + |\chi| = k$*

$$q_i(\varphi, \chi)(v_i \otimes 1)^{\otimes k} = (-1)^k v \left( 0, \dots, 0, \underbrace{\chi}_i, \underbrace{\varphi}_{i+1}, 0, \dots, 0 \right)$$

*Proof.*

$$\begin{aligned}
q_i(\varphi, \chi)(v_i \otimes 1)^{\otimes k} &= \Delta^{k-1}(q_i(\varphi, \chi))(v_i \otimes 1)^{\otimes k} \\
&= \left( \sum_{\substack{\psi \in \text{comp}_k(\chi) \\ \phi \in \text{comp}_k(\varphi)}} q_i(\phi(1), \psi(1)) \otimes \dots \otimes q_i(\phi(k), \psi(k)) \right) (v_i \otimes 1)^{\otimes k} \\
&\quad \text{by Lemma 3.1.2} \\
&= \sum_{\substack{\psi \in \text{comp}_k(\chi) \\ \phi \in \text{comp}_k(\varphi)}} (q_i(\phi(1), \psi(1))(v_i \otimes 1)) \otimes \dots \otimes (q_i(\phi(k), \psi(k))(v_i \otimes 1))
\end{aligned}$$

By Lemma 3.1.3 we see that the only potentially nonzero terms in the sum are those for which  $|\phi(j)| + |\psi(j)| \leq 1$  for all  $j \in \{1, \dots, k\}$ . Since  $|\varphi| + |\chi| = k$  if we have  $|\psi(j)| + |\phi(j)| = 0$  for some  $j \in \{1, \dots, n-1\}$  then there is a  $r \in \{1, \dots, n-1\}$  such that  $|\psi(r)| + |\phi(r)| > 1$ . So the only potentially nonzero terms in the sum are those for which  $|\phi(j)| + |\psi(j)| = 1$  for all  $j \in \{1, \dots, k\}$ . Suppose that  $\phi(j) = \chi_a$  and  $\psi(j) = 0$  for some  $j \in \{1, \dots, k\}$  and some  $a \in A$ . Then

$$q_i(\chi_a, 0)(v_i \otimes 1) = -(x_{-i} \otimes a)(v_i \otimes 1) = -(v_{i+1} \otimes a)$$

Suppose that  $\phi(j) = 0$  and  $\psi(j) = \chi_a$  for some  $j \in \{1, \dots, k\}$  and some  $a \in A$ . Then

$$q_i(0, \chi_a)(v_i \otimes 1) = -(h_i \otimes a)(v_i \otimes 1) = -(v_i \otimes a)$$

So  $-(v_{i+1} \otimes a)$  and  $-(v_i \otimes a)$  are the only possibilities for factors in the tensor product above. Since we are summing over all possible submultisets of  $\varphi$  and  $\chi$  we have the result. □

**Lemma 3.1.5.** *For all  $m \in \mathbb{N}$  and all  $\varphi_1, \dots, \varphi_n \in \mathcal{F}$  with  $\sum_{i=1}^n |\varphi_i| = m$*

$$q(\varphi_1, \dots, \varphi_n)(v_1 \otimes 1)^{\otimes m} = (-1)^{\sum_{j=1}^n j|\varphi_j|} v(\varphi_1, \dots, \varphi_n)$$

*Proof.* Since for all  $j \in \{1, \dots, n-1\}$  and  $k \in \{1, \dots, n\}$

$$x_{-j}v_k = \delta_{j,k}v_{j+1}, \quad h_jv_k = \delta_{j,k}v_j - \delta_{j+1,k}v_{j+1}$$

by Lemma 3.1.4 we have

$$q(\varphi_1, \dots, \varphi_n)(v_1 \otimes 1)^{\otimes m}$$

$$\begin{aligned} &= q_{n-1}(\varphi_n, \varphi_{n-1})q_{n-2}((|\varphi_n| + |\varphi_{n-1}|)\chi_1, \varphi_{n-2}) \dots q_1 \left( \left( \sum_{j=2}^n |\varphi_j| \right) \chi_1, \varphi_1 \right) (v_1 \otimes 1)^{\otimes m} \\ &= (-1)^m q_{n-1}(\varphi_n, \varphi_{n-1}) \dots q_2 \left( \left( \sum_{j=3}^n |\varphi_j| \right) \chi_1, \varphi_2 \right) v \left( \varphi_1, \left( \sum_{j=2}^n |\varphi_j| \right) \chi_1, 0, \dots, 0 \right) \\ &= (-1)^{|\varphi_1|+2\sum_{j=2}^n |\varphi_j|} q_{n-1}(\varphi_n, \varphi_{n-1}) \dots q_3 \left( \left( \sum_{j=4}^n |\varphi_j| \right) \chi_1, \varphi_3 \right) v \left( \varphi_1, \varphi_2, \left( \sum_{j=3}^n |\varphi_j| \right) \chi_1, 0, \dots, 0 \right) \\ &= (-1)^{\sum_{j=1}^{n-2} j|\varphi_j|} q_{n-1}(\varphi_n, \varphi_{n-1}) v(\varphi_1, \dots, \varphi_{n-2}, (|\varphi_{n-1}| + |\varphi_n|)\chi_1, 0) \\ &= (-1)^{\sum_{j=1}^n j|\varphi_j|} v(\varphi_1, \dots, \varphi_n) \end{aligned}$$

□

### 3.2. The Proof of Theorem 3.0.2.

*Proof.* By Lemmas 3.1.5 and 2.4.2

$$\left\{ q(\varphi_1, \dots, \varphi_n)(v_1 \otimes 1)^{\otimes m} \mid \varphi_1, \dots, \varphi_n \in \mathcal{F}(\mathbf{B}), \sum_{i=1}^n |\varphi_i| = m \right\}$$

is a basis for  $S^m(V \otimes A)$ . Therefore by Theorem 2.4.1

$$\left\{ q(\varphi_1, \dots, \varphi_n)w_{m\omega_1} \mid \varphi_1, \dots, \varphi_n \in \mathcal{F}(\mathbf{B}), \sum_{i=1}^n |\varphi_i| = m \right\}$$

is a basis for  $W_A(m\omega_1)$ .

□

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